

# AMOEBAS AND COAMOEBAS OF LINEAR SPACES

MOUNIR NISSE AND MIKAEL PASSARE

**ABSTRACT.** We give a complete description of the amoebas and coamoebas of  $k$ -dimensional affine linear spaces in  $(\mathbb{C}^*)^n$ . We give a lower and an upper bounds of their dimension. and we show that if a  $k$ -dimensional affine linear space in  $(\mathbb{C}^*)^n$  is generic, then the dimension of its (co)amoeba is equal to  $\min\{2k, n\}$ . We also prove that the volume of its coamoeba is equal  $\pi^{2k}$ . Moreover, if the space is real then the volume of its amoeba is equal to  $\frac{\pi^{2k}}{2^k}$ .

## 1. INTRODUCTION

Amoebas and coamoebas are very fascinating notions in mathematics, the first terminology has been introduced by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky in 1994 [GKZ-94], and the second by the second author in a talk in 2004. Amoebas and coamoebas, which are natural projections of complex varieties, and which turn out to have relations to several other fields: tropical geometry, real algebraic geometry, generalized hypergeometric functions, mirror symmetry, and others (e.g., [M1-04], [M2-00], [NS1-11], [PR-04], [SS-04], and others). More precisely, the amoebas (resp. coamoebas) of complex algebraic and analytic varieties of the complex algebraic torus  $(\mathbb{C}^*)^n$  are defined as their image under the logarithmic mapping  $\text{Log} : (z_1, \dots, z_n) \mapsto (\log |z_1|, \dots, \log |z_n|)$  (resp. the argument mapping  $\text{Arg} : (z_1, \dots, z_n) \mapsto (e^{i \arg(z_1)}, \dots, e^{i \arg(z_n)})$ ). Amoebas (resp. Coamoebas) are the link between classical complex algebraic geometry and tropical (resp. complex tropical) geometry. More precisely, amoebas degenerate to piecewise-linear objects called tropical varieties (see [M1-04], and [PR-04]), and coamoebas degenerate to a non-Archimedean coamoebas which are the coamoebas of some lifting in the complex algebraic torus of tropical varieties. (See [NS2-11] for more details about non-Archimedean coamoebas, and [N2-09], [N3-11] about this degeneration in case of hypersurfaces.)

(Co)Amoebas, being of a logarithmic nature, it is natural that they are closely related to the exponents of the defining functions of  $V$ , and to the associated Newton polytopes. This connection is extensively explored in the thesis of the first author [N1-10]. Another important connection is to the currently very active field of tropical geometry, a piecewise linear incarnation of classical algebraic geometry where the varieties can be seen as non-Archimedean versions of amoebas.

Among the results obtained in this paper, one can mention Theorem 2.1 and Theorem 3.1 where the dimension of the (co)amoeba of an affine linear

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space is determined in terms of the dimension of the complex space itself. Theorem 5.1 states that the (co)amoeba of an algebraic variety  $V$  is equal to the intersection of all hypersurface (co)amoebas corresponding to functions in the defining ideal  $\mathcal{I}(V)$  of  $V$ . This is a fundamental result for the general study of amoebas and coamoebas that do not come from hypersurfaces. Some of these results are obtained by the first author in his thesis [N1-10].

It was shown by the first author and F. Madani in [MN1-11] that the volume of the amoeba of a generic  $k$ -dimensional algebraic variety of  $(\mathbb{C}^*)^n$  with  $n \geq 2k$  is finite. Moreover, they proved in [MN2-11] that the finiteness of the volume of the amoeba of a generic analytic variety is equivalent to the fact that the variety is algebraic. In Theorem 6.1, we compute explicitly the volume of the amoeba and the coamoeba of a generic real  $k$ -dimensional affine linear space of  $(\mathbb{C}^*)^{2k}$ . In particular, we show that the volume of the coamoebas of complex and real  $k$ -dimensional affine linear spaces of  $(\mathbb{C}^*)^{2k}$  are equal.

**Remark.** This work was started on June 2011, but after the tragic death of Mikael Passare on 15 September 2011, the completion and writing of this paper was done by the first author.

## 2. AMOEBAS OF AFFINE LINEAR SPACES

Let  $k$ , and  $m$  be two positives natural integers, and  $\mathcal{P}(k) \subset (\mathbb{C}^*)^{k+m}$  be the  $k$ -dimensional affine linear space given by the parametrization  $\rho$  as follows:

$$\rho : \begin{array}{ccc} (\mathbb{C}^*)^k & \longrightarrow & (\mathbb{C}^*)^{k+m} \\ (t_1, \dots, t_k) & \longmapsto & (t_1, \dots, t_k, f_1(t_1, \dots, t_k), \dots, f_m(t_1, \dots, t_k)), \end{array} \quad (1)$$

where  $f_j(t_1, \dots, t_k) = b_j + \sum_{i=1}^k a_{ji}t_i$ , and  $a_{ji}$ ,  $b_j$  are complex numbers for  $i = 1, \dots, k$ , and  $j = 1, \dots, m$ . First of all, we remark that if  $\mathcal{P}(k)$  is generic then there exists  $j$  such that  $b_j \neq 0$ . Indeed, otherwise there is an action of  $\mathbb{C}^*$  on  $\mathcal{P}(k)$ , and then it can be seen as a product of  $\mathbb{C}^*$  with an affine linear space of dimension  $k-1$ . So, we can assume that  $f_1(t_1, \dots, t_k) = 1 + \sum_{i=1}^k t_i$ . In fact, assume  $b_1 \neq 0$ , so, we can make a translation by  $\frac{1}{b_1}$  in the algebraic torus  $(\mathbb{C}^*)^{k+m}$  (multiplicative group) to get

$$\left( \frac{t_1}{b_1}, \dots, \frac{t_k}{b_1}, \frac{f_1(t_1, \dots, t_k)}{b_1}, \dots, \frac{f_m(t_1, \dots, t_k)}{b_1} \right).$$

Then, we translate by  $a = (a_{11}, a_{21}, \dots, a_{k1}, 1, \dots, 1)$  to obtain:

$$\left( \frac{a_{11}t_1}{b_1}, \dots, \frac{a_{k1}t_k}{b_1}, 1 + \sum_{i=1}^k \frac{a_{i1}t_i}{b_1}, \frac{f_2(t_1, \dots, t_k)}{b_1}, \dots, \frac{f_m(t_1, \dots, t_k)}{b_1} \right).$$

If we denote by  $\psi$  the required parametrization, i.e.,

$$\psi(t_1, \dots, t_k) = (t_1, \dots, t_k, 1 + \sum_{i=1}^k t_i, f_2(t_1, \dots, t_k), \dots, f_m(t_1, \dots, t_k)),$$

then we get  $\tau_a \circ \tau_{\frac{1}{b_1}} \circ \rho = \psi \circ \tau_c$ , where  $c = (\frac{a_{11}}{b_1}, \dots, \frac{a_{k1}}{b_1})$ . Hence, for any  $(t_1, \dots, t_k)$  in  $(\mathbb{C}^*)^k$  we have:

$$\text{Arg}(\rho(t_1, \dots, t_k)) - \text{Arg}(b_1) + \text{Arg}(a) = \text{Arg}(\psi(\tau_c(t_1, \dots, t_k))),$$

and we have the same relation if we replace the argument map by the logarithmic map. This means that the amoeba (resp. coamoeba) of the complex affine linear space  $\mathcal{P}(k)$  given by the parametrization  $\rho$  is the translation in the real space  $\mathbb{R}^{k+m}$  (resp. the real torus  $(S^1)^{k+m}$ ) by a vector in  $\mathbb{R}^{k+m}$  (resp. a point in the torus) depending on the logarithm of  $b_1$  and  $a$  (resp. their arguments) of an affine linear space given by a parametrization such that  $f_1(t_1, \dots, t_k) = 1 + \sum_{i=1}^k t_i$ . So,  $\text{co}\mathcal{A}(\mathcal{P}(k)) = \tau_v \circ \text{co}\mathcal{A}(\mathcal{P}_\psi(k))$  where  $\mathcal{P}_\psi(k)$  is the affine linear space given by the required parametrization, and we have a similar equality for the amoeba. In the last formula,  $v$  and  $v'$  are the arguments (resp. the logarithm) of the vector  $b_1^{-1}a$ . Hence, we can assume in all this section that  $f_1(t_1, \dots, t_k) = 1 + \sum_{i=1}^k t_i$ , and for all  $j = 2, \dots, m$ ,  $f_j(t_1, \dots, t_k) = b_j + \sum_{i=1}^k a_{ji}t_i$ . Moreover, we assume that the affine linear spaces are in general position (i.e., not contained in a subspace parallel to the axis or a linear subspace of  $(\mathbb{C}^*)^{k+m}$ ).

**Theorem 2.1.** *Let  $k$ , and  $m$  be two positives natural integers, and  $\mathcal{P}(k) \subset (\mathbb{C}^*)^{k+m}$  be an affine linear space of dimension  $k$ . Then, the dimension of the amoeba  $\mathcal{A}_k$  of  $\mathcal{P}(k)$  satisfies the following:*

$$\dim(\mathcal{A}_k) \leq \min\{2k, k + m\}.$$

*In particular, if  $\mathcal{P}(k)$  is in general position, then the dimension of its amoeba is equal to  $\min\{2k, k + m\}$ .*

*Proof.* We can check that if  $k = 1$ , and  $\mathcal{P}(1)$  is generic, then the dimension is always equal two. We give an explicit equations and inequalities satisfied by the semi-algebraic subset  $\mathcal{S}_k$  of  $\mathbb{R}^{k+m}$  such that its image under the logarithmic map is the amoeba of our affine linear space. Let us fixe the following notations:

- (i) For any  $t = (t_1, \dots, t_k) \in (\mathbb{C}^*)^k$  and  $j = 1, \dots, m$ , we set  $y_j = |f_j(t)|$ , and for any  $i = 1, \dots, k$  we denote by  $r_i = |t_i|$  and  $\theta_i = \arg(t_i)$ ;
- (ii) For any coefficient  $a_{ji}$  and  $b_j$  of the affine linear forms  $f_j$ , we put  $\theta_{a_{ji}} = \arg(a_{ji})$  and  $\theta_{b_j} = \arg(b_j)$ .

With the above notations, using the fact that we can assume that  $f_1$  is real, and with coefficients equal to one, we obtain:

$$\begin{cases} y_1^2 &= 1 + \sum_{i=1}^k r_i^2 + 2 \sum_{i=1}^k r_i \cos \theta_i + 2 \sum_{i \neq l} r_i r_l \cos(\theta_i - \theta_l), \\ y_j^2 &= |b_j|^2 + \sum_{i=1}^k |a_{ji}|^2 r_i^2 + 2 \sum_{i=1}^k |b_j| |a_{ji}| r_i \cos(\theta_i + \theta_{a_{ji}} - \theta_{b_j}) \\ &\quad + 2 \sum_{i \neq l} |a_{ji}| |a_{jl}| r_i r_l \cos((\theta_i - \theta_l) + (\theta_{a_{ji}} - \theta_{a_{jl}})) \end{cases}$$

for any  $j = 2, \dots, m$ . It is clear that the amoeba  $\mathcal{A}_k$  is contained in the image under the logarithmic map of the semi-algebraic set  $\mathcal{S}_k$  parametrized by:

$$(r_1, \theta_1, r_2, \theta_2, \dots, r_k, \theta_k) \mapsto (r_1, r_2, \dots, r_k, y_1, y_2, \dots, y_m).$$

More precisely,  $\mathcal{A}_k$  is the image under the logarithmic map of some closed subset  $\mathcal{S}'_k$  of  $\mathcal{S}_k$  with nonempty interior. So, the dimension of the amoeba is equal to that of  $\mathcal{S}_k$  because the logarithmic map restricted to the positive

quadrant (i.e., to  $(\mathbb{R}_+^*)^{k+m}$ ) is a diffeomorphism. Hence, it cannot exceed the minimum of  $2k$  and the dimension of the ambient space  $k+m$ , because the parameters here are the  $r_i$ 's and the  $\theta_i$ 's.

$$\begin{array}{ccccc}
 \mathcal{P}(k) & \xrightarrow{i} & (\mathbb{C}^*)^{k+m} & & \\
 \downarrow |\cdot| & & \downarrow |\cdot| & & \\
 \mathcal{S}'_k & \xrightarrow{i} & \mathcal{S}_k & \xrightarrow{i} & (\mathbb{R}_+^*)^{k+m} \\
 \downarrow \text{Log} & & \downarrow \text{Log} & & \downarrow \text{Log} \\
 \mathcal{A}_k & \xrightarrow{i} & M_k & \xrightarrow{i} & \mathbb{R}^{k+m}.
 \end{array}$$

where the horizontal maps (i.e., the  $i$ 's) are the canonical inclusions. The subset  $\mathcal{S}'_k$  is given by the  $m \times (k+1)$  triangular inequalities of the  $(k+1)$ -plane coordinates. If  $\mathcal{P}(k)$  is in general position in the sense that it is not parallel to some  $k$ -plane axes and it is not a  $k$ -plane passing through the origin, then the dimension is maximal. Indeed, let  $\mathcal{P}(k+1)$  in general position in  $(\mathbb{C}^*)^{m+k+1}$ . So, the  $k$ -plane  $\mathcal{P}(k) \subset \mathcal{P}(k+1)$  defined by the same parametrization with fixed  $t_{k+1}$  (i.e., the last coordinate) is in general position. By induction on  $k$ , the dimension in this case viewed as subspace of  $(\mathbb{C}^*)^{m+k} \subset (\mathbb{C}^*)^{m+k} \times \{t_{k+1}\} \subset (\mathbb{C}^*)^{m+k+1}$  is equal to  $\min\{2k, k+m\}$ . If the minimum is equal to  $2k$  then the number of constraints is  $m-k$ , but there is another degree of freedom due to the argument of  $t_{k+1}$ . Hence, the dimension is  $m+k+1 - (m-k) + 1 = 2(k+1)$ . If  $m+k < 2k$ , then it is obvious that the minimum of  $2(k+1)$  and  $m+k+1$  is  $m+k+1$ .  $\square$

### 3. COAMOEBAS OF AFFINE LINEAR SPACES

In this section we prove a similar theorem for coamoebas as in Section 1. Using the same notations as above we obtain the following:

**Theorem 3.1.** *Let  $k$ , and  $m$  be two positives natural integers, and  $\mathcal{P}(k) \subset (\mathbb{C}^*)^{k+m}$  be an affine linear space of dimension  $k$ . Then, the dimension of the coamoeba  $\text{co}\mathcal{A}_k$  of  $\mathcal{P}(k)$  satisfies the following:*

$$\dim(\text{co}\mathcal{A}_k) \leq \min\{2k, k+m\}.$$

*In particular, if  $\mathcal{P}(k)$  is in general position, then the dimension of its coamoeba is equal to  $\min\{2k, k+m\}$ .*

*Proof.* Let  $\mathcal{P}(k) \subset (\mathbb{C}^*)^{k+m}$  be the affine linear space of dimension  $k$  given by the parametrization  $\rho$ :

$$\begin{aligned}
 \rho : (\mathbb{C}^*)^k & \longrightarrow (\mathbb{C}^*)^{k+m} \\
 (t_1, \dots, t_k) & \longmapsto (t_1, \dots, t_k, f_1(t_1, \dots, t_k), \dots, f_m(t_1, \dots, t_k)),
 \end{aligned}$$

such that  $f_1(t_1, \dots, t_k) = 1 + \sum_{i=1}^k t_i$ , and  $f_j(t_1, \dots, t_k) = b_j + \sum_{i=1}^k a_{ji} t_i$  for  $j = 2, \dots, m$  where  $a_{ji}$  and  $b_j$  are complex numbers for  $i = 1, \dots, k$ . Using the notations in Section 2, and the fact that  $\frac{t}{t} = \frac{t^2}{tt}$  for any complex number  $t$  different than zero. If we denote by  $\psi_j = \arg(f_j(t))$ , we get the following:

$$\frac{f_j(t)}{f_j(t)} = \frac{b_j}{b_j} \left( \frac{1 + \sum_{i=1}^k \frac{a_{ji} t_i}{b_j}}{1 + \sum_{i=1}^k \frac{a_{ji} t_i}{b_j}} \right) \quad \text{for } j = 2, \dots, m,$$

if  $b_j \neq 0$ , and

$$\frac{f_j(t)}{f_j(t)} = \frac{\sum_{i=1}^k a_{ji} t_i}{\sum_{i=1}^k a_{ji} t_i}$$

otherwise. Hence, if  $b_j \neq 0$  (We do the computation in this case only, the other case can be done by the same method.)

$$2\psi_j = \arg\left(\frac{f_j(t)}{f_j(t)}\right) = 2\theta_{b_j} + 2\arg\left(1 + \sum_{i=1}^k \frac{a_{ji} t_i}{b_j}\right) \mod (2\pi).$$

Then,

$$\begin{cases} \psi_1 &= \arg(1 + \sum_{i=1}^k t_i) \mod (\pi) \\ \psi_j &= \theta_{b_j} + \arg(1 + \sum_{i=1}^k \frac{a_{ji} t_i}{b_j}) \mod (\pi) \end{cases}$$

for  $j = 2, \dots, m$ , and then,

$$\begin{cases} \psi_1 &= \arctan\left(\frac{\sum_{i=1}^k r_i \sin(\theta_i)}{1 + \sum_{i=1}^k r_i \cos(\theta_i)}\right) \mod (\pi) \\ \psi_j &= \theta_{b_j} + \arctan\left(\frac{\sum_{i=1}^k |\frac{a_{ji}}{b_j}| r_i \sin(\theta_i + \theta_{a_{ji}} - \theta_{b_j})}{1 + \sum_{i=1}^k |\frac{a_{ji}}{b_j}| r_i \cos(\theta_i + \theta_{a_{ji}} - \theta_{b_j})}\right) \mod (\pi) \end{cases}$$

for  $j = 2, \dots, m$ .

This gives a parametrization of the coamoeba  $co\mathcal{A}_k$  of the complex linear space  $\mathcal{P}(k)$  given by the parametrization  $\rho$ . It is clear now, that the dimension of the image of  $(\mathbb{C}^*)^k$  by the argument map satisfies the conclusion of Theorem 2.1.  $\square$

#### 4. AMOEBAS OF AFFINE LINES IN $(\mathbb{C}^*)^{1+m}$

We give in this section a complete description of the amoebas of complex lines in  $(\mathbb{C}^*)^{1+m}$ . We describe also, the real lines, i.e., the lines parametrized as above such that the coefficients  $a_j$  and  $b_j$  are real for any  $j = 1, \dots, m$ . First of all, let us assume that the coefficients are complex. By the parametrization (1) of Section 2, where  $f_1(t) = 1 + t$ , if we denote by  $r = |t|$  and  $\theta = \arg(t)$ , we get:

$$\begin{cases} y_1^2 &= 1 + r^2 + 2r \cos \theta \\ y_j^2 &= |b_j|^2 + |a_j|^2 r^2 + 2|b_j||a_j|r \cos(\theta + \theta_{a_j} - \theta_{b_j}) \end{cases}$$

for any  $j = 2, \dots, m$ .

Hence we have:

$$y_j^2 = |b_j|^2 + |a_j|^2 r^2 + 2|b_j||a_j|r \cos(\theta) \cos(\theta_{a_j} - \theta_{b_j}) - 2|b_j||a_j|r \sin(\theta) \sin(\theta_{a_j} - \theta_{b_j}).$$

We denote by:

$$\begin{cases} T_j &= |b_j||a_j|(y_1^2 - r^2 - 1) \cos(\theta_{a_j} - \theta_{b_j}) \\ W_j &= -y_j^2 + |a_j|^2 r^2 + |b_j|^2. \end{cases}$$

**Lemma 4.1.** *There are two types of amoebas of lines in  $(\mathbb{C}^*)^{m+1}$  for  $m \geq 2$ . Amoebas with boundary and other without boundary. Using the notation as above, the amoebas are with boundary if and only if  $\sin(\theta_{a_j} - \theta_{b_j}) = 0$ , which is equivalent to saying that  $[\frac{a_2}{b_2} : \dots : \frac{a_m}{b_m}] \in \mathbb{RP}^{m-2}$ .*

*Proof.* Using the fact that

$$2r \cos \theta = y_1^2 - r^2 - 1 \text{ and } 2r \sin \theta = \sqrt{4r^2 - (y_1^2 - r^2 - 1)^2},$$

we get:

$$(W_j + T_j)^2 = |b_j|^2 |a_j|^2 (4r^2 - (y_1^2 - r^2 - 1)) \sin^2(\theta_{a_j} - \theta_{b_j}).$$

Hence, if  $\sin(\theta_{a_j} - \theta_{b_j}) = 0$  which is equivalent to saying that  $[1 : \frac{a_2}{b_2} : \dots : \frac{a_m}{b_m}] \in \mathbb{RP}^{m-1}$  (i.e., the line is real). Then, the last expression can be written as follows:

$$W_j + T_j = 0, \tag{2}$$

for all  $j = 2, \dots, m$ .  $\square$

This brings us to the following definition:

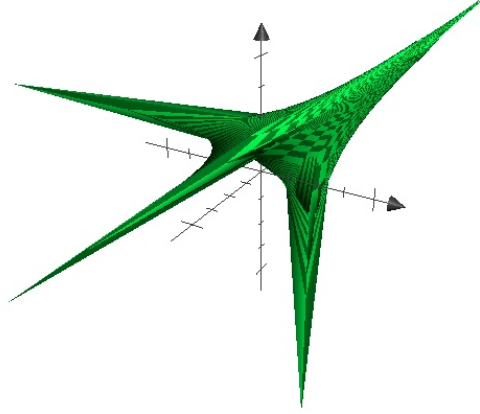
**Definition 4.1.** *A  $k$ -dimensional affine linear space given by the parametrization  $\rho$  in (1) is called real if and only if  $[\frac{a_{1i}}{b_{1i}} : \dots : \frac{a_{mi}}{b_{mi}}] \in \mathbb{RP}^{m-1}$  for all  $i = 1, \dots, k$ .*

**Remark 4.1.** *We can check, using the first remark in Section 2, that if the  $k$ -dimensional affine linear space given by the parametrization  $\rho$  in (1) is real, then, up to a translation in the complex algebraic torus, all the coefficient in the parametrization  $\rho$  can be assumed real*

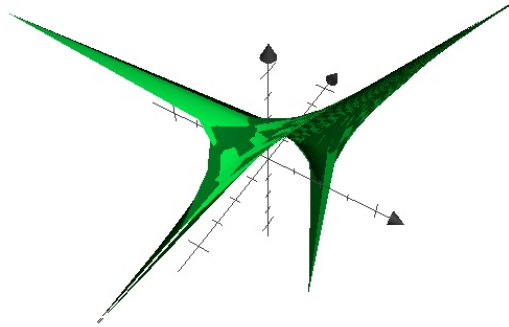
**Example 4.1.** *Consider the real line  $L$  in  $(\mathbb{C}^*)^3$  given by the parametrization  $t \mapsto (t, t+1, -2t+5)$ . Recall that  $m = 2$ ; so, using (1) we obtain the equation of the quadric in  $\mathbb{R}^3$  given by  $y_2^2 + 10y_1^2 - 14r^2 - 35 = 0$  such that its intersection with the quadrant  $(\mathbb{R}_+^*)^3$  is a surface  $\mathcal{S}_1$  diffeomorphic to  $\mathbb{R}^2$  (recall that the coordinates of  $\mathbb{R}^3$  are denoted by  $(r, y_1, y_2)$ ). The image of  $\mathcal{S}_1$  under the logarithmic map is a surface  $M_1$  containing the amoeba  $\mathcal{A}_1$  of  $L$ .*

## 5. (CO)AMOEBAS OF COMPLEX ALGEBRAIC VARIETIES

We describe the amoeba (resp. coamoeba) of a complex variety  $V$  with defining ideal  $\mathcal{I}(V)$  as the intersection of the amoebas (resp. coamoebas) of the complex hypersurfaces with defining polynomials in  $\mathcal{I}(V)$ . More precisely, we have the following theorem (see [N1-10]):



**Figure 1.** The amoeba of the complex line in  $(\mathbb{C}^*)^3$  given by the parametrization  $t \mapsto (t, t+1, t-2i)$ . Topologically it is a sphere without four points.



**Figure 2.** The amoeba of the real line in  $(\mathbb{C}^*)^3$  parametrized by  $t \mapsto (t, t+1, t-2)$ . Topologically it is a closed disk without four points in its boundary.

**Theorem 5.1.** *Let  $V \subset (\mathbb{C}^*)^n$  be an algebraic variety with defining ideal  $\mathcal{I}(V)$ . Then the amoeba (resp. coamoeba) of  $V$  is given as follows:*

$$\mathcal{A}(V) = \bigcap_{f \in \mathcal{I}(V)} \mathcal{A}(V_f) \quad (\text{resp. } \text{co}\mathcal{A}(V) = \bigcap_{f \in \mathcal{I}(V)} \text{co}\mathcal{A}(V_f)).$$

The part concerning the amoeba of this theorem is a corollary of some result of Kevin Purbhoo [P-08]. But we present here, a very simple proof.

*Proof.* Let  $V_f \subset (\mathbb{C}^*)^n$  be a hypersurface with defining polynomial  $f$ . Then, by definition, the amoeba of  $V_f$  is the image by the logarithmic map of the

subset  $\mathcal{S}_f$  of  $(\mathbb{R}_+^*)^n$  defined as follows:

$$\mathcal{S}_f := \{(x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n \mid x_i = |z_i|, \text{ and } f(z_1, \dots, z_n) = 0\}.$$

The real logarithmic map is a diffeomorphism between  $(\mathbb{R}_+^*)^n$  and  $\mathbb{R}^n$ , this implies the following:

$$\bigcap_{f \in \mathcal{I}(V)} \text{Log}(\mathcal{S}_f) = \text{Log} \left( \bigcap_{f \in \mathcal{I}(V)} \mathcal{S}_f \right).$$

But we have the following:

$$\bigcap_{f \in \mathcal{I}(V)} \mathcal{S}_f = \bigcap_{f \in \mathcal{I}(V)} \{(x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n \mid x_i = |z_i|, \text{ and } f(z_1, \dots, z_n) = 0\}.$$

So, it remains to prove that the last set is the same as the following set:

$$\{(x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n \mid (x_1, \dots, x_n) = (|z_1|, \dots, |z_n|) \text{ and } z \in V\}.$$

**Lemma 5.1.** *We have the following equality:*

$$\bigcap_{f \in \mathcal{I}(V)} \mathcal{S}_f = \{(x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n \mid x_i = |z_i|, \text{ and } (z_1, \dots, z_n) \in V\}.$$

*Proof.* Let  $r$  be in

$$(\mathbb{R}_+^*)^n \setminus \{(x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n \mid x_i = |z_i| \text{ and } (z_1, \dots, z_n) \in V\},$$

and  $T_r$  be the real torus  $\text{Log}^{-1}(\text{Log}(r))$ . So,  $T_r \cap V$  is empty. Let  $f \in \mathcal{I}(V)$  with  $f(z) = \sum a_\alpha z^\alpha$  and  $g$  be the Laurent polynomial defined by  $g(z) = \sum \bar{a}_\alpha w^\alpha$  with  $w = (\frac{r_1^2}{z_1^2}, \dots, \frac{r_n^2}{z_n^2})$  where the  $r_j$ 's are the coordinates of  $r$ , and  $\bar{a}_\alpha$  denotes the conjugate of the coefficient  $a_\alpha$ . It is clear that the Laurent polynomial  $h(z) = f(z)g(z)$  is equal to  $|f(z)|^2$  in the torus  $T_r$ . By construction, the hypersurface  $V_h$  with defining polynomial  $h$  contains  $V$  (because  $h \in \mathcal{I}(V)$ ). Let  $\langle f_1, \dots, f_s \rangle$  be a set of generator of the ideal  $\mathcal{I}(V)$ , and for any  $j$  let  $g_j$  be the Laurent polynomial defined as before. We can verify immediately that the hypersurface defined by the polynomial  $G = \sum f_j g_j$  contains  $V$  and don't intersect the torus  $T_r$ . This proves that  $r \in (\mathbb{R}_+^*)^n \setminus \bigcap_{f \in \mathcal{I}(V)} \mathcal{S}_f$ . Hence, we have the inclusion:

$$\bigcap_{f \in \mathcal{I}(V)} \mathcal{S}_f \subset \{(x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n \mid x_i = |z_i|, \text{ and } (z_1, \dots, z_n) \in V\}.$$

The other inclusion is obvious. Hence,  $\text{Log} \left( \bigcap_{f \in \mathcal{I}(V)} \mathcal{S}_f \right) = \mathcal{A}(V)$ , and then

$$\mathcal{A}(V) = \bigcap_{f \in \mathcal{I}(V)} \mathcal{A}(V_f).$$

□

*End of the proof of Theorem 4.2.* The first inclusion of the second equality is obvious. Let  $w \in \bigcap_{f \in \mathcal{I}(V)} \text{co}\mathcal{A}(V_f)$ , then there exists a fundamental domain  $\mathcal{D} = [a; a + 2\pi]^n$  in the universal covering of the real torus  $(S^1)^n$  and a unique  $\tilde{w} \in \mathcal{D}$  such that  $w = \exp(i\tilde{w})$ . In this domain, the exponential map is a diffeomorphism between  $\mathcal{D}$  and  $(S^1)^n \setminus S^1 \wedge \dots \wedge S^1$  where  $S^1 \wedge \dots \wedge S^1$



denotes the bouquet of  $n$  circles. Let us define the subset  $co\mathcal{S}_f$  of  $\mathcal{D}$  as follow:

$$co\mathcal{S}_f := \{\theta \in \mathcal{D} \mid \text{there exists } z \in V_f \text{ and } \exp(i\theta) = \text{Arg}(z)\}.$$

So, we have:

$$\bigcap_{f \in \mathcal{I}(V)} \exp(ico\mathcal{S}_f) = \exp\left(i \bigcap_{f \in \mathcal{I}(V)} co\mathcal{S}_f\right)$$

because the exponential map is a diffeomorphism from  $\mathcal{D}$  into its image. Moreover,  $\tilde{w}$  is in the intersection  $\bigcap_{f \in \mathcal{I}(V)} co\mathcal{S}_f$ . But the last intersection, using the same argument of Lemma 4.3, can be described as follow:

$$\begin{aligned} \bigcap_{f \in \mathcal{I}(V)} co\mathcal{S}_f &= \bigcap_{f \in \mathcal{I}(V)} \{\theta \in \mathcal{D} \mid \text{there exists } z \in V_f \text{ and } \exp(i\theta) = \text{Arg}(z)\} \\ &= \{\theta \in \mathcal{D} \mid \text{there exists } z \in V \text{ and } \exp(i\theta) = \text{Arg}(z)\} \end{aligned}$$

Indeed, let  $e^{i\theta} \notin co\mathcal{A}(V)$ , and for each generator  $f_j(z) = \sum a_\alpha z^\alpha$  of  $\mathcal{I}(V)$  we define the polynomial  $g_j$  as follows:

$$g_j(z) = \sum \bar{a}_\alpha (e^{-2i\theta})^\alpha z^\alpha.$$

It is clear that over  $\text{Arg}(e^{i\theta})$  we have  $f_j g_j(z) = |f_j(z)|^2$ , and then the polynomial  $G = \sum_j f_j g_j$  is in  $\mathcal{I}(V)$  but  $e^{i\theta} \notin co\mathcal{A}(V_G)$ . This means that we have the following inclusion:

$$\bigcap_{f \in \mathcal{I}(V)} co\mathcal{S}_f \subset \{\theta \in \mathcal{D} \mid \text{there exists } z \in V \text{ and } \exp(i\theta) = \text{Arg}(z)\}.$$

The other inclusion is obvious. □

## 6. VOLUME OF (CO)AMOEBAS OF $k$ -DIMENSIONAL AFFINE LINEAR SPACES OF $(\mathbb{C}^*)^{2k}$

It was shown by the first author and F. Madani in [MN1-11] that the volume of the amoeba of a generic  $k$ -dimensional algebraic variety in  $(\mathbb{C}^*)^n$  with  $n \geq 2k$  is finite. In this section, we compute the volume of a generic real  $k$ -dimensional affine linear space in  $(\mathbb{C}^*)^{2k}$ . So, we suppose in this section that  $\mathcal{P}(k)$  is a generic affine linear subspace of  $(\mathbb{C}^*)^{2k}$ , where the genericity means that the Jacobian of the logarithmic map restricted to  $\mathcal{P}(k)$  is of maximal rank in an open subset of  $\mathcal{P}(k)$ . We prove the following:

**Theorem 6.1.** *Let  $\mathcal{P}(k)$  be a generic affine linear subspace of  $(\mathbb{C}^*)^{2k}$ . Then we have the following:*

- (i) *The volume of the coamoeba  $co\mathcal{A}(\mathcal{P}(k))$  is equal to  $\pi^{2k}$ ;*
- (ii) *Moreover, if  $\mathcal{P}(k)$  is real, then the volume of its amoeba  $\mathcal{A}(\mathcal{P}(k))$  is equal to  $\frac{\pi^{2k}}{2^k}$ .*

Let  $\mathcal{P}(k) \subset (\mathbb{C}^*)^{2k}$  be a generic  $k$ -dimensional affine linear space. Suppose that  $\mathcal{P}(k)$  is given by the parametrization  $\rho$ :

$$\begin{aligned} \rho : (\mathbb{C}^*)^k &\longrightarrow (\mathbb{C}^*)^{2k} \\ (t_1, \dots, t_k) &\longmapsto (t_1, \dots, t_k, f_1(t_1, \dots, t_k), \dots, f_k(t_1, \dots, t_k)), \end{aligned}$$

such that  $f_j(t_1, \dots, t_k) = b_j + \sum_{i=1}^k a_{ji} t_i$  where  $a_{ji}$ , and  $b_j$  are complex numbers for  $i = 1, \dots, k$  and  $j = 1, \dots, k$ . The space  $\mathcal{P}(k)$  is generic, then there exists at least  $j \in \{1, \dots, k\}$  such that  $b_j \neq 0$ . Indeed, let  $z \in \mathcal{P}(k)$ , and  $L_z$  be the line passing by the origin and containing  $z$ . So,  $L_z$  is linear, and contained in  $\mathcal{P}(k)$ . But the tangent space to  $L_z$  at  $z$  contains a purely imaginary vector, this means that  $z$  is a critical point of the logarithmic mapping restricted to  $L_z$ . Hence,  $z$  is also a critical point of the logarithmic mapping restricted to  $\mathcal{P}(k)$  (see [MN3-12]). This implies that, for any point  $z \in \mathcal{P}(k)$  the Jacobian of the logarithmic map restricted to  $\mathcal{P}(k)$  is not of maximal rank. So,  $\mathcal{P}(k)$  is not generic.

Let  $\mathbb{Z}_2 := \{\pm 1\} \subset \mathbb{C}^*$ , and  $\mathbb{Z}_2^{2k} \subset (\mathbb{C}^*)^{2k}$ . For each  $s \in \mathbb{Z}_2^{2k}$ , let  $\mathcal{P}_s(k)$  be the  $k$ -dimensional affine linear space of  $(\mathbb{C}^*)^{2k}$  parametrized by  $\rho_s(t_1, \dots, t_k) = s \cdot \rho(t_1, \dots, t_k)$ .

**Proposition 6.1.** *With the above notations, we have the following statements:*

- (i) *The argument map from the subset of regular points of  $\mathcal{P}_s(k)$  to the subset of regular values of its coamoeba  $\text{co}\mathcal{A}(\mathcal{P}_s(k))$  is injective;*
- (ii) *Let  $s \neq r$  in  $\mathbb{Z}_2^{2k}$ , and denote by  $\text{Reg}(\text{co}\mathcal{A}(\mathcal{P}_u(k)))$  the subset of regular values of the coamoeba  $\text{co}\mathcal{A}(\mathcal{P}_u(k))$  for  $u \in \mathbb{Z}_2^{2k}$ , then:*

$$\text{Reg}(\text{co}\mathcal{A}(\mathcal{P}_s(k))) \cap \text{Reg}(\text{co}\mathcal{A}(\mathcal{P}_r(k)))$$

*is empty;*

- (iii) *The union  $\bigcup_{s \in \mathbb{Z}_2^{2k}} \text{Reg}(\text{co}\mathcal{A}(\mathcal{P}_s(k)))$  is an open dense subset of the real torus  $(S^1)^{2k}$ .*

*Proof.* Let  $\Theta = (e^{i\theta_1}, \dots, e^{i\theta_k}, e^{i\psi_1}, \dots, e^{i\psi_k})$  be a point in the set of regular values of  $\text{co}\mathcal{A}(\mathcal{P}(k))$ . This is equivalent to the fact that the following linear system  $(E)$  of  $2k$  equations and  $2k$  variables  $(x_1, \dots, x_k, y_1, \dots, y_k)$  in  $(\mathbb{R}_+^*)^{2k}$ :

$$\begin{cases} \text{Re}(b_j + \sum_{l=1}^k a_{jl} x_l e^{i\theta_l}) &= \text{Re}(y_j e^{i\psi_j}) \\ \text{Im}(b_j + \sum_{l=1}^k a_{jl} x_l e^{i\theta_l}) &= \text{Im}(y_j e^{i\psi_j}) \end{cases}$$

with  $j = 1 \dots, k$ , has a solution. Moreover, if  $\mathbb{Z}_2^{2k}$  is viewed as a subgroup of the real torus  $(S^1)^{2k}$ , then saying that  $s \cdot \Theta \in \bigcup_{s \in \mathbb{Z}_2^{2k}} \text{Reg}(\text{co}\mathcal{A}(\mathcal{P}_s(k)))$  is equivalent to saying that the system  $(E)$  has a solution in  $(\mathbb{R}^*)^{2k}$ .

Since the matrix  $A(z)$  defined by:

$$A(z) = \begin{pmatrix} a_{11}z_1 & a_{12}z_2 & \dots & a_{1k}z_k & -z_{k+1} & 0 & 0 & \dots & 0 \\ a_{21}z_1 & a_{22}z_2 & \dots & a_{2k}z_k & 0 & -z_{k+2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1}z_1 & a_{k2}z_2 & \dots & a_{kk}z_k & 0 & 0 & 0 & \dots & -z_{2k} \end{pmatrix}$$

is precisely the image under the logarithmic Gauss map of the point  $z$ . Hence,  $A(z)$  is of rank  $k$  when  $z$  is a regular point. Also, for any regular point  $z$ , the matrix  $\hat{A}(z) = \begin{pmatrix} A(z) \\ \bar{A}(z) \end{pmatrix}$  is of rank  $2k$  (see [MN3-12]). The matrix defining the real system  $(E)$  is precisely  $\tilde{B}(\Theta) = \begin{pmatrix} \operatorname{Re} B(\Theta) \\ \operatorname{Im} B(\Theta) \end{pmatrix}$  where  $B(\Theta)$  is the following matrix:

$$B(\Theta) = \begin{pmatrix} a_{11}e^{i\theta_1} & a_{12}e^{i\theta_2} & \dots & a_{1k}e^{i\theta_k} & -e^{i\psi_1} & 0 & 0 & \dots & 0 \\ a_{21}e^{i\theta_1} & a_{22}e^{i\theta_2} & \dots & a_{2k}e^{i\theta_k} & 0 & -e^{i\psi_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1}e^{i\theta_1} & a_{k2}e^{i\theta_2} & \dots & a_{kk}e^{i\theta_k} & 0 & 0 & 0 & \dots & -e^{i\psi_k} \end{pmatrix}$$

We can check that the real rank of  $\tilde{B}(\Theta)$  is the same as the rank of the matrix  $\tilde{A}(z) = \begin{pmatrix} \operatorname{Re} A(z) \\ \operatorname{Im} A(z) \end{pmatrix}$  with  $z = (x_1e^{i\theta_1}, \dots, x_ke^{i\theta_k}, y_1e^{i\psi_1}, \dots, y_ke^{i\psi_k})$ .

We claim that the real rank of the matrix  $\tilde{A}(z)$  is  $2k$ . Indeed, on the contrary, there exist a real numbers  $\lambda_l$ , and  $\mu_l$  not all equal to zero, with  $l = 1, \dots, k$  such that:

$$\sum_{l=1}^k \frac{\lambda_l}{2} \left( \sum_{j=1}^k (z_j a_{lj} + \bar{z}_j \bar{a}_{lj}) - (z_{k+l} + \bar{z}_{k+l}) \right) + \sum_{l=1}^k \frac{\mu_l}{2i} \left( \sum_{j=1}^k (z_j a_{lj} - \bar{z}_j \bar{a}_{lj}) - (z_{k+l} - \bar{z}_{k+l}) \right) = 0.$$

We get:

$$\sum_{l=1}^k \left( \frac{\lambda_l - i\mu_l}{2} \right) \left( \sum_{j=1}^k z_j a_{lj} - z_{k+l} \right) + \sum_{l=1}^k \left( \frac{\lambda_l + i\mu_l}{2} \right) \left( \sum_{j=1}^k \bar{z}_j \bar{a}_{lj} - \bar{z}_{k+l} \right) = 0.$$

Since the matrix  $\hat{A}(z)$  is of rank  $2k$ , this implies that  $\lambda_l - i\mu_l = 0$ , and  $\lambda_l + i\mu_l = 0$  for all  $l = 1, \dots, k$ , which means that all the  $\lambda_l$ 's, and the  $\mu_l$ 's must vanished. Hence, the real rank of the matrix  $\tilde{A}(z)$  is  $2k$ . This shows that the real linear system  $(E)$  has a unique solution for any  $\Theta$  in the set of regular values of  $\operatorname{co}\mathcal{A}(\mathcal{P}(k))$ , which proves the first two statements of the proposition. The third statement comes from the fact that the distance between any point in the complement of the coamoeba  $\operatorname{co}\mathcal{A}(\mathcal{P}(k))$ , and the coamoeba itself cannot exceed  $\pi$ .  $\square$

**Corollary 6.1.** *The volume of the coamoeba of any generic  $k$ -dimensional linear space in  $(\mathbb{C}^*)^{2k}$  is equal to  $\pi^{2k}$ .*

*Proof.* The volume of the disjoint union  $\bigcup_{s \in \mathbb{Z}_2^{2k}} \operatorname{Reg}(\operatorname{co}\mathcal{A}(\mathcal{P}_s(k)))$  is equal to the volume of all the real torus  $(S^1)^{2k}$ . Moreover, they have the same volume, because they are obtained from each other by an isometry. So, the volume of one of them must be equal to  $(2\pi)^{2k}/2^{2k} = \pi^{2k}$ .  $\square$

It was shown by the first author with F. Madani the following two facts. Let  $V \subset (\mathbb{C}^*)^n$  be an algebraic variety, and  $\mathcal{L}og$  denotes a branch of a holomorphic logarithm  $(z_1, \dots, z_n) \mapsto (\log(z_1), \dots, \log(z_n))$  defined in a neighborhood of a point  $z$  in  $V$ . It was shown by the first author with F. Madani the following two lemmas (see [MN3-12]).

**Lemma 6.1** (Madani-Nisse). *Let  $V \subset (\mathbb{C}^*)^n$  be a  $k$ -dimensional algebraic variety, and  $z$  be a smooth point of  $V$ . Then  $z$  is a critical point for the map  $\text{Log}|_V$  if and only if the image of the tangent space  $T_z V$  to  $V$  at  $z$  by  $d\mathcal{L}og$  contains at least  $s$  purely imaginary vectors with  $s = \max\{1, 2k - n + 1\}$ .*

**Proposition 6.2** (Madani-Nisse). *Let  $\mathcal{P}(k) \subset (\mathbb{C}^*)^n$  be a generic  $k$ -dimensional affine linear space with  $n \geq 2k$ . Suppose that the complex dimension of  $\mathcal{P}(k) \cap \overline{\mathcal{P}(k)}$  is equal to  $l$ , with  $0 \leq l \leq k$ . Then, for any regular value  $x$  in the amoeba  $\mathcal{A}(\mathcal{P}(k))$ , the cardinality of  $\text{Log}^{-1}(x)$  is at least  $2^l$ .*

Using Lemma 6.1, and Proposition 6.2, we obtain the following:

**Proposition 6.3.** *Let  $\mathcal{P}(k)$  be a generic real affine linear subspace of  $(\mathbb{C}^*)^{2k}$ , and  $x$  be a regular value of its amoeba. Then, the cardinality of  $\text{Log}^{-1}(x)$  is equal  $2^k$ .*

*Proof.* Let  $\mathcal{P}(k) \subset (\mathbb{C}^*)^{2k}$  be a generic  $k$ -dimensional real affine linear space. So, we can assume that  $\mathcal{P}(k)$  is given by a parametrization  $\rho$  as before, where all the coefficients are real numbers. The matrix  $A$  defined by:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}$$

is invertible, otherwise the image of  $\rho$  is a linear space of dimension strictly less than  $k$ . The following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{C}^*)^k & \xrightarrow{\rho} & (\mathbb{C}^*)^{2k} \\ A \downarrow & & \downarrow A \times Id \\ (\mathbb{C}^*)^k & \xrightarrow{\rho'} & (\mathbb{C}^*)^{2k} \end{array}$$

where  $\rho'$  is the parametrization given by:

$$\begin{aligned} \rho' : (\mathbb{C}^*)^k &\longrightarrow (\mathbb{C}^*)^{2k} \\ (T_1, \dots, T_k) &\longmapsto (T_1, \dots, T_k, b_1 + T_1, \dots, b_k + T_k). \end{aligned}$$

It is clear that each regular value of the amoeba of the  $k$ -dimensional linear space  $\mathcal{L}(k) = \rho'((\mathbb{C}^*)^k)$  is covered  $2^k$  times under the logarithmic mapping. Indeed, it is a product of a plane lines  $L_1, \dots, L_k$ . The matrix  $A$  is real, so the image of the set of critical points of the logarithmic mapping restricted to  $\mathcal{P}(k)$  is the set of critical points of the logarithmic mapping restricted to  $\mathcal{L}(k)$ . Indeed, if  $z$  is a critical in  $\mathcal{P}(k)$ , then the tangent space to  $\mathcal{P}(k)$  at  $z$  contains a purely imaginary vector  $v_z$ . The image of  $v_z$  in the tangent space to  $\mathcal{L}(k)$  at  $(A \times Id)(z)$  is also purely imaginary tangent vector, and then, the point  $(A \times Id)(z)$  is critical. This means that the number of connected components of  $\mathcal{P}(k) \setminus \text{Critp}(\text{Log}|_{\mathcal{P}(k)})$  is equal to the number of connected

components of  $\mathcal{L}(k) \setminus \text{Critp}(\text{Log}|_{\mathcal{L}(k)})$ . By Proposition 6.1, the restriction of the argument map to the set of regular points is injective. Hence, the cardinality of  $\text{Log}^{-1}(x)$  is at most  $2^k$ . (We used the fact that the set of critical points of the argument, and the logarithmic maps coincide, e.g., see [MN2-11].) Since  $\mathcal{P}(k)$  is real, then by Proposition 6.2, for any regular value  $x \in \mathcal{A}(\mathcal{P}(k))$ , the cardinality of  $\text{Log}^{-1}(x)$  is at least  $2^k$ . Hence, it is equal to  $2^k$ .  $\square$

The second statement of Theorem 6.1 is an immediate consequence of Proposition 6.1, and Proposition 6.3. The particular case of 2-dimensional affine real linear space of  $(\mathbb{C}^*)^4$  was computed by P. Johansson using trigonometric relations (see [J-10]).

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368, USA.

E-mail address: [nisse@math.tamu.edu](mailto:nisse@math.tamu.edu)

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, SE-106 91 STOCKHOLM, SWEDEN

E-mail address: [passare@math.su.se](mailto:passare@math.su.se)